ON THE STABILITY OF MOTION OF A BODY IN A FLUID

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Stability of steady helical motions of a rigid body in a fluid was investigated by Liapunov [1]. Here the problem is considered for the case when the body has a cavity completely filled with a viscous fluid. Results obtained by Rumiantsev in [2] are utilized in this investigation.

1. We shall consider the motion of a mechanical system represented by a rigid body with a cavity completely filled with a viscous incompressible fluid, in the infinite perfect fluid undergoing an irrotational motion and at rest at infinity. We assume that the only force acting on the body and the fluids under consideration is that of gravity and that the weight of displaced fluid is equal to that of the body together with the viscous fluid.

We shall use a rectangular coordinate system $Ox_1x_2x_3$ with the origin at the center of mass of the body and the axes and their unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ coinciding with the principal central axes of inertia of the body relative to the point O. Then, the equations of motion of our mechanical system will be (1.1)

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{U}} + \boldsymbol{\omega}\times\frac{\partial T}{\partial \mathbf{U}} = 0, \quad \frac{d}{dt}\frac{\partial T}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega}\times\frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{U}\times\frac{\partial T}{\partial \mathbf{U}} = Mg\boldsymbol{\gamma}\times\mathbf{r}_0, \quad \frac{d\boldsymbol{\gamma}}{dt} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = 0$$

$$\frac{d}{dt}(\mathbf{U} + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{u}) + \boldsymbol{\omega}\times(\mathbf{U} + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{u}) = -g\boldsymbol{\gamma} - \frac{1}{\rho}\operatorname{grad} p + \nu\Delta\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0$$

where $T = T^{(1)} + T^{(2)} + T^{(3)}$ is the kinetic energy of the system. $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are the kinetic energies of the body, the viscous fluid and the perfect fluid, respectively, while $\mathbf{U}$ and $\boldsymbol{\omega}$ are the respective vectors of the translational and instantaneous angular velocity of the body,

$$\frac{\partial T}{\partial \mathbf{U}} = \frac{\partial T}{\partial U_1}\mathbf{i}_1 + \frac{\partial T}{\partial U_2}\mathbf{i}_2 + \frac{\partial T}{\partial U_3}\mathbf{i}_3, \qquad \frac{\partial T}{\partial \boldsymbol{\omega}} = \frac{\partial T}{\partial \omega_1}\mathbf{i}_1 + \frac{\partial T}{\partial \omega_2}\mathbf{i}_2 + \frac{\partial T}{\partial \omega_3}\mathbf{i}_3$$

are the momentum and angular momentum vectors respectively, referred to the point O: $\boldsymbol{\gamma}$ is a unit vector in the vertical direction: $\mathbf{r}_0$ is a vector produced from the center of gravity of the volume bounded by the surface of the body adjacent to the infinite fluid, to the center of mass of the body and the fluid inside its cavity: $M = M^{(1)} + M^{(2)}$, where $M^{(1)}$ and $M^{(2)}$ denote the mass of the body and the viscous fluid respectively: g is the acceleration due to gravity; $\mathbf{u}$ is the vector of the relative velocity of particles of the viscous fluid: p and ρ is the pressure and density of the fluid respectively: $\nu = \mu / \rho$ is the kinematic viscosity coefficient: μ is the coefficient of viscosity and Δ is the Laplace operator.

Equations (1.1) require an additional condition on the walls S of the cavity, and this condition is : $\mathbf{u} = 0$ on S.

From (1.1) we obtain

$$\frac{d}{dt}(T+V) = -\mu\int_\tau (\nabla \times \mathbf{u})^2\, d\tau \qquad (V = Mg\mathbf{r}_0\boldsymbol{\gamma})$$

where V is the potential energy of the system, ∇ is the Hamiltonian operator and τ is the volume of the cavity. From this,

$$T + V \leqslant T_0 + V_0 \qquad (T_0 = T\,|_{t=0},\ V_0 = V\,|_{t=0}) \tag{1.2}$$

follows.

Equations (1.1) also admit the following integrals :

$$\left(\frac{\partial T}{\partial \mathbf{U}}\right)^2 = n^2 = \text{const}, \qquad \frac{\partial T}{\partial \mathbf{U}}\,\boldsymbol{\gamma} = \text{const}, \qquad \boldsymbol{\gamma}^2 = 1 \tag{1.3}$$

$$\frac{\partial T}{\partial \boldsymbol{\omega}}\,\boldsymbol{\gamma} = \text{const} \qquad \text{for} \quad \mathbf{U} = 0 \tag{1.4}$$

$$\frac{\partial T}{\partial \mathbf{U}}\,\frac{\partial T}{\partial \boldsymbol{\omega}} = \text{const} \qquad \text{for} \quad \mathbf{r}_0 = 0 \tag{1.5}$$

2. To simplify the calculations we shall assume that the centers of mass of the body and the fluid contained in its cavity coincide, and, that their principal axes of inertia relative to their common center of mass, also coincide. Then we have the following expressions fo $T^{(1)}$ and $T^{(2)}$

$$T^{(1)} = \frac{1}{2} M_1\mathbf{U}^2 + \frac{1}{2}\,\boldsymbol{\omega}\theta^{(1)}\boldsymbol{\omega}$$

$$T^{(2)} = \frac{1}{2} M_2\mathbf{U}^2 + \frac{1}{2}\,\boldsymbol{\omega}\theta^{(2)}\boldsymbol{\omega} + \boldsymbol{\omega}\mathbf{g} + \frac{1}{2}\rho\int_\tau \mathbf{u}^2\, d\tau, \qquad \mathbf{g} = \rho\int_\tau \mathbf{r}\times\mathbf{u}\, d\tau$$

Here $\mathbf{g}$ is the vector of relative angular momentum of the viscous fluid, while $\theta^{(1)}$ and $\theta^{(2)}$ are the inertia tensors of the body and the viscous fluid for the point O

$$\theta_{ii}^{(1)} = I_i, \quad \theta_{ii}^{(2)} = J_i, \quad \theta_{ij}^{(1)} = \theta_{ij}^{(2)} = 0 \quad (i \neq j,\ i, j = 1, 2, 3)$$

The following expression [3] shall be used for $T^{(3)}$

$$T^{(3)} = \frac{1}{2}\sum_{i=1}^{3} (A_iU_i^2 + 2B_iU_i\omega_i + C_i\omega_i^2)$$

where A_i, B_i and C_i are known constants.

Following [2] (p. 137), we shall introduce the vectors $\boldsymbol{\Omega}(t)$ and $\mathbf{u}_*(t, r)$

$$\boldsymbol{\omega}\theta^{(2)} + \mathbf{g} = \boldsymbol{\Omega}\theta^{(2)}, \qquad \boldsymbol{\omega}\times\mathbf{r} + \mathbf{u} = \boldsymbol{\Omega}\times\mathbf{r} + \mathbf{u}_*$$

Expressing $T^{(2)}$ in the form

$$T^{(2)} = \frac{1}{2} M_2\mathbf{U}^2 + \frac{1}{2}(\boldsymbol{\omega}\theta^{(2)} + \mathbf{g})\,\theta^{(2)-1}(\boldsymbol{\omega}\theta^{(2)} + \mathbf{g}) + \frac{1}{2}\rho\int_\tau \mathbf{u}_*^2\, d\tau$$

we readily obtain the inequality

$$2T^{(2)} \geqslant M_2\mathbf{U}^2 + (\boldsymbol{\omega}\theta^2 + \mathbf{g})^2 J^{-1}, \qquad J = \max(J_1, J_2, J_3)$$

analogous to that given in [2] (p. 137).

3. Let, during the whole motion, $\mathbf{U} = 0$. Then, when $\mathbf{r}_0 = x_{30}\mathbf{i}_3$ the equations (1.1) admit a particular solution

$$\boldsymbol{\omega} = \omega\mathbf{i}_3, \quad \boldsymbol{\gamma} = \mathbf{i}_3, \quad \mathbf{g} = 0, \quad \mathbf{u} = 0 \tag{3.1}$$

We shall investigate the stability of motion (3.1) with respect to $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ and $\mathbf{g}$. Replacing these magnitudes in (1.2) to (1.4) with their perturbed values and denoting the left-hand sides of these integrals by $V_1, \ldots, V_5$ respectively, we shall consider the function

$$V = V_1 + \sigma V_4 - 2\omega V_5 + {}^1/_4 \lambda V_4^2 \qquad (\sigma = (C_3 + I_3 + J_3)\,\omega^2 - Mgx_{30},\ \lambda > Mgx_{30})$$

which, provided the conditions

$$(C_3 + I_3 + J_3 - C_i - I_i - J_i)\,\omega^2 - Mgx_{30} > 0 \qquad (i = 1, 2)$$

are fulfilled, satisfies ([2], pp. 138-139) all the conditions of the Rumiantsev theorem on the stability with respect to the part of the variables, which in turn proves the stability of the unperturbed motion (3.1) with respect to the above mentioned magnitudes. In addition, the stability with respect to

$$\rho \int_\tau u_{i_*}^2\, d\tau \qquad (i = 1, 2, 3)$$

is also obvious.

4. When $\mathbf{r}_0 = 0$, Equations (1.1) admit a particular solution

$$\mathbf{U} = U\mathbf{i}_3, \quad \boldsymbol{\omega} = \omega\mathbf{i}_3, \quad \boldsymbol{\gamma} = \mathbf{i}_3, \quad \mathbf{g} = 0, \quad \mathbf{u} = 0 \tag{4.1}$$

Let us investigate the stability of motion (4.1) with respect to the magnitudes $\mathbf{U}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ and $\mathbf{g}$.

Replacing in (1.2), (1.3) and (1.5) $\mathbf{U}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$, $\mathbf{g}$ and $\mathbf{u}$ with their perturbed values and denoting the left-hand sides of these integrals by $V_1, \ldots, V_4$ and V_6 respectively, we shall consider the function

$$V = V_1 + \lambda V_2 + \sigma V_3 + \varkappa V_4 + \mu V_6 + {}^1/_4 \nu V_4^2$$

where

$$\sigma = 2\,[(A_3 + M)\,U + B_3\omega]^{-1}\,\{(C_3 + I_3 + J_3)\,\omega^2 - (A_3 + M)\,U^2 - \\ - \lambda\,[(A_3 + M)\,U + B_3\omega]^2\}$$

$$\varkappa = (A_3 + M)\,U^2 - (C_3 + I_3 + J_3)\,\omega^2 + \lambda\,[(A_3 + M)\,U + B_3\omega]^2$$

$$\mu = -2\omega\,[(A_3 + M)\,U + B_3\omega]^{-1}$$

while λ and ν are sufficiently small positive constants.

Limiting ourselves to the case when $B_1 = B_2 = B_3 = 0$, we obtain the following sufficient conditions of stability of unperturbed motion (4.1) with respect to the above mentioned magnitudes

$$(A_3 + M)\,(A_3 + M_2 - A_i)\,U^2 + (A_i + M_1)\,(C_3 + I_3 + J_3 - C_i - I_i)\,\omega^2 > 0 \quad (i = 1, 2)$$

$$(A_3 + M)\,(A_3 - A_i)\,U^2 + (A_i + M)\,(C_3 + I_3 + J_3 - C_i - I_i - J_i)\,\omega^2 > 0$$

5. Assuming that the initial impulse of the system has a vertical axis for both, perturbed and unperturbed motion, let us replace $\boldsymbol{\gamma}$ in (1.1) with $(\varepsilon / n)\,\partial T / \partial \mathbf{U}$, where $\epsilon = -1$ if the direction of the impulse coincides with the direction of the force of gravity and $\epsilon = +1$ otherwise, and n is a constant appearing in the first of the integrals of (1.3) and is assumed to remain the same for both, perturbed and unperturbed motion. Then, for $\mathbf{r}_0 = x_{30}\mathbf{i}_3$ Equations (1.1) admit the following particular solution

$$\mathbf{U} = U\mathbf{i}_3, \quad \boldsymbol{\omega} = \omega\mathbf{i}_3, \quad \mathbf{g} = 0, \quad \mathbf{u} = 0 \tag{5.1}$$

Let us investigate the stability of this unperturbed motion (5.1) with respect to **U**, **ω** and **g**. We replace **γ** in (1.2) with $(\varepsilon / n)\, \partial T / \partial \mathbf{U}$; **U**, **ω**, **g** and **u** in (1.2), in the first of the integrals of (1.3) and in (1.5) which now holds for any $\mathbf{r}_0$ with their perturbed values, we denote the left-hand sides of these integrals by V_1, V_2 and V_3, respectively, and we consider the function

$$V = V_1 + \lambda V_2 + \varkappa V_3 + \nu V_2^2, \qquad \varkappa = \frac{-2\omega}{(A_3 + M)\, U}$$

where

$$\lambda = \frac{[(C_3 + I_3 + J_3)\, \omega^2 - (A_3 + M)\, U\, (U + U_*)]}{(A_3 + M)^2\, U^2}, \qquad U_* = \frac{\varepsilon}{n} Mgx_{30}$$

and ν is a sufficiently large positive constant.

Limiting ourselves to the case when $B_1 = B_2 = B_3 = 0$, we obtain the following sufficient conditions of stability of the unperturbed motion (5.1) with respect to the magnitudes shown above $(i = 1,2)$

$$(A_3 + M)(A_3 + M_2 - A_i)\, U^2 + (A_i + M_1)[(C_3 + I_3 + J_3 - C_i - I_i)\, \omega^2 - \varepsilon\, Mgx_{30}] > 0$$

$$(A_3 + M)\, (A_3 - A_i)\, U^2 + (A_i + M)\, [(C_3 + I_3 + J_3 - C_i - I_i - J_i)\, \omega^2 - \varepsilon Mgx_{30}] > 0$$

It should be noted that these conditions are, by virtue of the limitations imposed at the beginning of this section, the sufficient conditions of conditional stability of the unperturbed motion (5.1) with respect to the indicated magnitudes.

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